Online Appendix to "Invoicing Currency and Optimal Policies in a Global Liquidity Trap"

A.1 Robustness Checks

We demonstrate that our key findings about the relationship between pricing regimes and optimal policy responses remain robust across different parameter specifications. We examine variations in three additional critical parameters from our baseline calibration: trade elasticity θ, inverse elasticity of intertemporal substitution ρ, and Calvo price stickiness α. These parameters have been identified as fundamental determinants of international shock transmission and policy effectiveness in the open economy literature (Benigno and Benigno, 2006; Corsetti et al., 2011). For each alternative parameter configuration, we analyze how optimal policy varies across both trade openness (ν) and shock magnitudes (σ_z), maintaining the same parameter space explored in Section 7 of the paper. This approach allows us to systematically verify that our core findings about the interaction between pricing regimes, trade integration, and shock size remain qualitatively robust to different parameter choices.

A particularly important composite parameter in open economy models is $\rho\theta$, which governs the substitutability between home and foreign goods. We systematically explore how optimal policy varies under three scenarios: when goods are Edgeworth substitutes ($\rho\theta > 1$), complements ($\rho\theta < 1$), and independent ($\rho\theta = 1$). We achieve these different scenarios through targeted adjustments to either ρ or θ while holding other parameters constant.

Figures A1 and A2 examine the case where $\rho = 0.5$, corresponding to higher intertemporal elasticity of substitution. Under this parameterization, home and foreign goods become Edgeworth independent ($\rho\theta = 1$). This independence weakens the efficacy of Foreign's monetary policy in facilitating expenditure switching or counteracting perverse exchange rate movements under large shocks. Consequently, Foreign maintains rates near zero even under PCP across a broader range of parameters (Figure A1). The higher intertemporal elasticity of substitution also makes consumption more responsive to real interest rate deviations, leading to more severe recessions. This necessitates substantially larger fiscal interventions in both world and relative terms (Figure A2), though the qualitative patterns of optimal policy remain consistent with our baseline findings.

Figures A3 and A4 consider the case with $\rho = 2$, featuring stronger Edgeworth substitutability between goods. While Foreign still raises rates under PCP to manage relative prices (Figures A3), these increases are less pronounced than in our baseline calibration. This moderated monetary response reflects the lower intertemporal elasticity of substitution, which dampens the severity of recessions and reduces the need for aggressive policy intervention. The world and relative fiscal gaps demonstrate similar qualitative patterns but smaller magnitudes, consistent with this reduced severity (Figure A4).

Figures A5 and A6 analyze the complementarity case by setting $\theta = 0.5$. Under Edgeworth complementarity, both PCP and LCP frameworks exhibit similar monetary policy responses, with Foreign maintaining zero rates across most parameter combinations (Figure A5). This convergence occurs because complementarity weakens the expenditure-switching channel: households consume home and foreign goods together rather than substituting between them. This both reduces the need to manage relative distortions and dampens global recession severity, explaining the smaller world and relative fiscal gaps compared to our baseline (Figure A6).

Figures A7 and A8 examine another case of Edgeworth independence by setting $\theta = 1$. While independence again reduces Foreign's incentive to raise rates under PCP (Figure A7), baseline intertemporal elasticity results in milder recessions than the $\rho = 0.5$ case, requiring smaller fiscal interventions (Figure A8).

Finally, Figures A9 and A10 examine the impact of reduced price stickiness ($\alpha = 0.75$). While the qualitative patterns of optimal monetary and fiscal policy remain unchanged, the magnitude of required interventions increases substantially. This amplification reflects the "Paradox of Flexibility" identified by Eggertsson and Woodford (2003) and Werning (2011), where greater price flexibility can exacerbate economic distortions at the zero lower bound by intensifying deflationary pressures. As can be seen from these robustness exercises, we found that the relationship between pricing regimes and optimal policy responses remains robust across different parameter specifications.

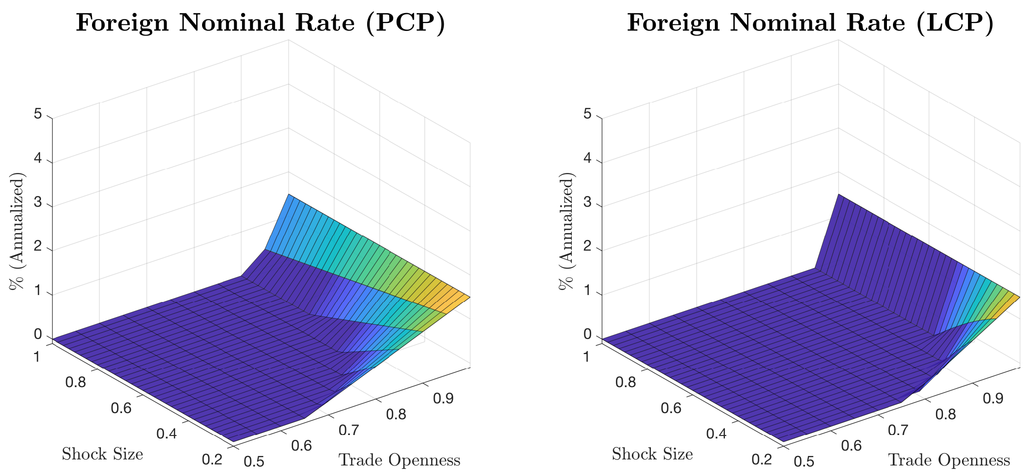

Figure A1: Trade Openness and Optimal Monetary Policy ($\rho = 0.5$)

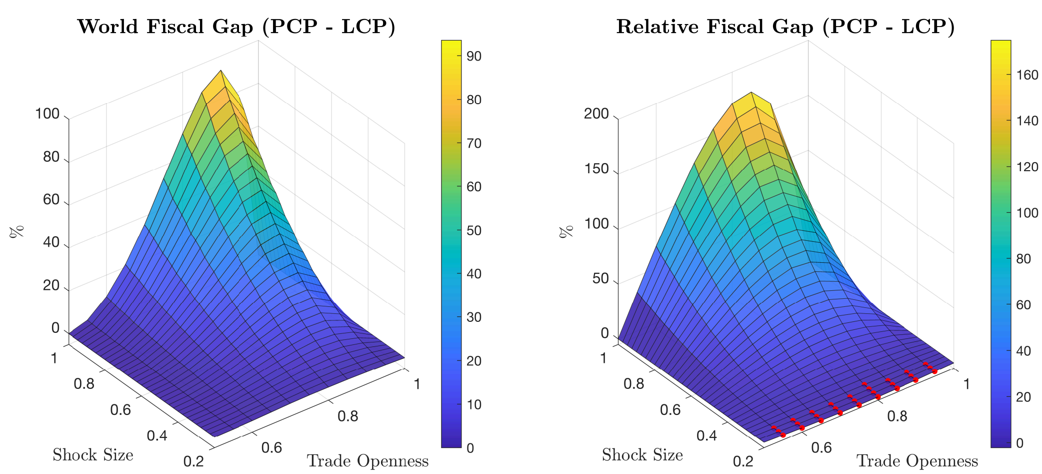

Figure A2: Trade Openness and Optimal Fiscal Policy ($\rho = 0.5$)

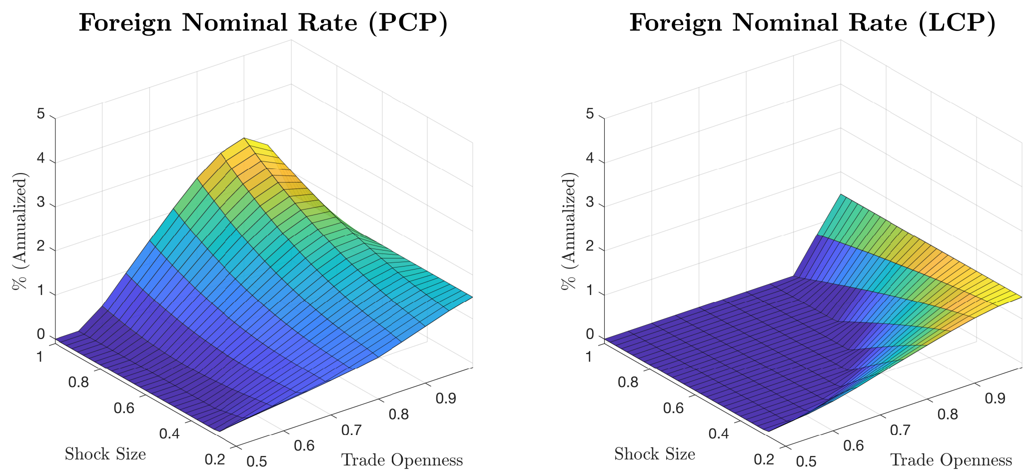

Figure A3: Trade Openness and Optimal Monetary Policy ($\rho = 2$)

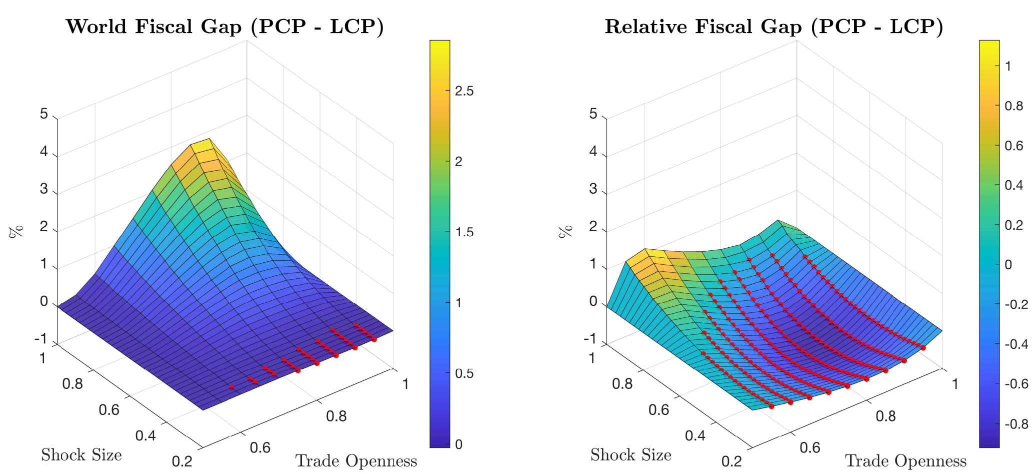

Figure A4: Trade Openness and Optimal Fiscal Policy ($\rho = 2$)

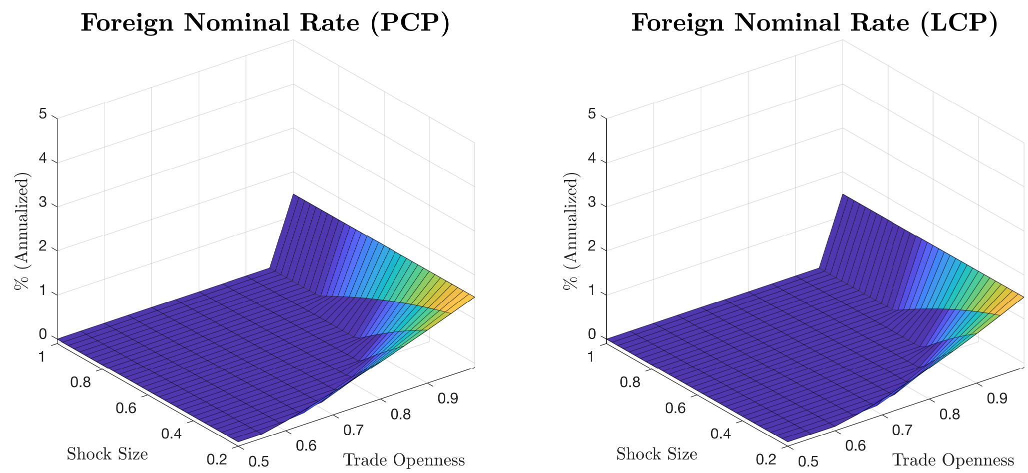

Figure A5: Trade Openness and Optimal Monetary Policy ($\theta = 0.5$)

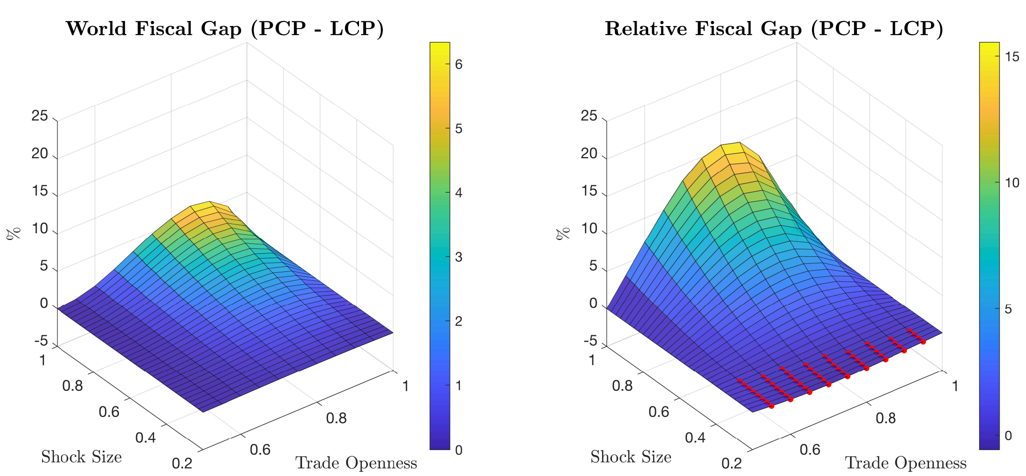

Figure A6: Trade Openness and Optimal Fiscal Policy ($\theta = 0.5$)

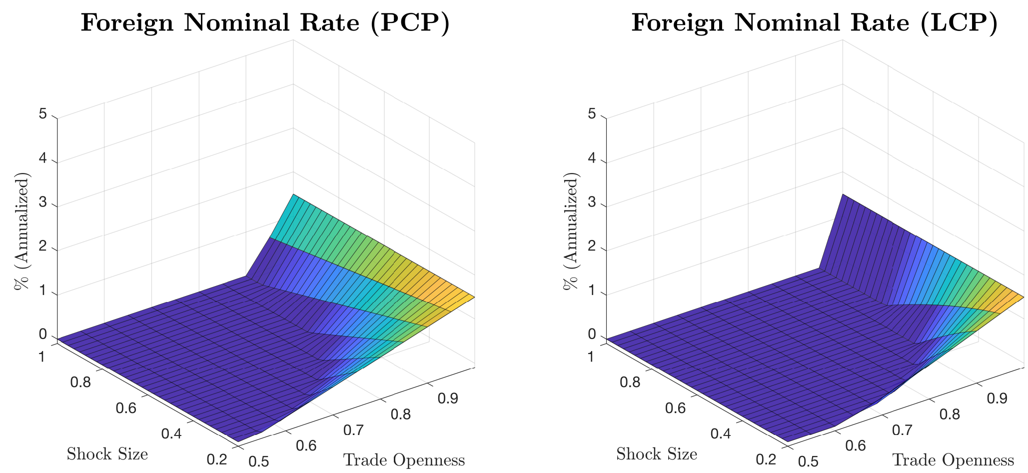

Figure A7: Trade Openness and Optimal Monetary Policy ($\theta = 1$)

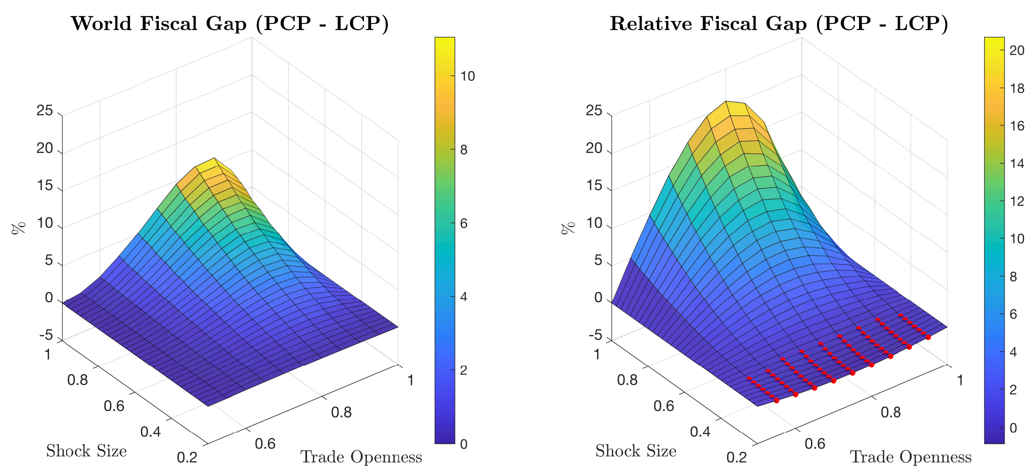

Figure A8: Trade Openness and Optimal Fiscal Policy ($\theta = 1$)

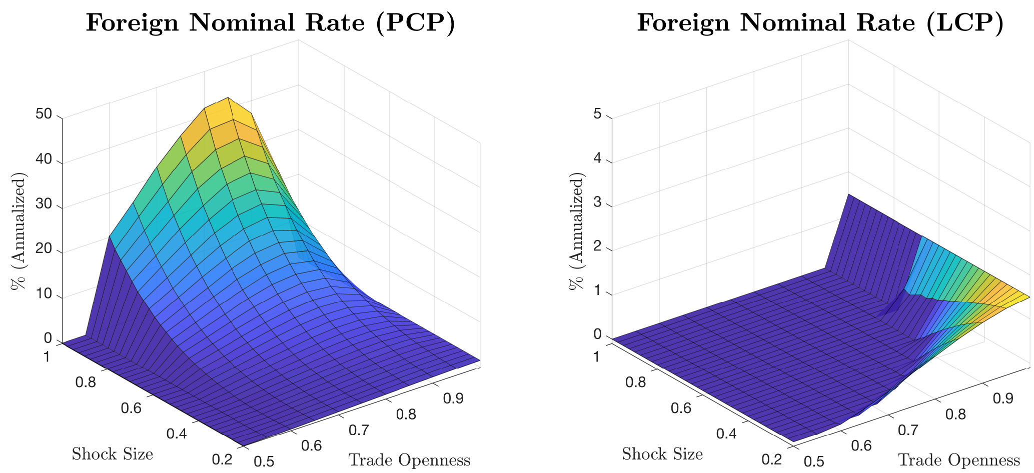

Figure A9: Trade Openness and Optimal Monetary Policy ($\alpha = 0.75$)

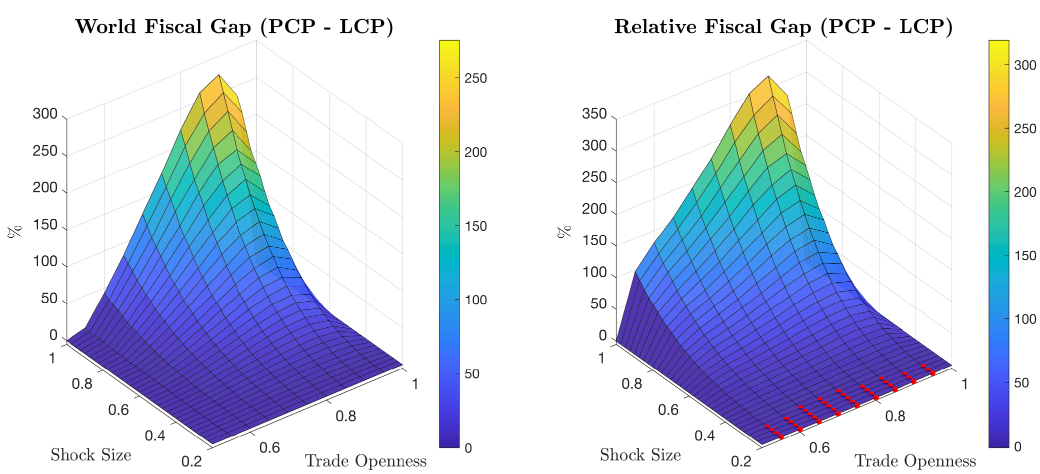

Figure A10: Trade Openness and Optimal Fiscal Policy ($\alpha = 0.75$)

B.1 A Two-Country Model with Various Pricing Regimes

This section presents the log-linearized equilibrium conditions necessary to define equilibrium under each pricing regime. The variables are specified for each country, Home and Foreign. The transformation of these variables into World and Relative terms follows the approach detailed in the appendix of Engel (2011). Since the first-best allocation is identical across the regime, we provide the equilibrium conditions only in Section B.2.1.

B.2 Log-linearized Equilibrium Conditions

B.2.1 Equilibrium conditions under PCP

Given the path of the four policy instruments $\{R_t, R_t^*, G_t, G_t^*\}$, the equilibrium under flexible exchange rates consists of endogenous variables $\{\hat{Y}_{H,t}, \hat{Y}_{F,t}, \hat{\pi}_{H,t}, \hat{\pi}^*_{F,t}, \hat{C}_t, \hat{C}_t^*, \hat{T}_t, \hat{Q}_t\}$ and exogenous variables $\{\hat{Z}_t, \hat{Z}_t^*\}$ that satisfy:

- Aggregate demand:

$$\mathbb{E}_t\hat{C}_{t+1} = \hat{C}_t + \frac{1}{\rho}\left\{\hat{R}_t - \left[\mathbb{E}_t\pi_{H,t+1} + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}_{t+1} - \hat{Z}_t\right\}$$
$$\mathbb{E}_t\hat{C}^*_{t+1} = \hat{C}^*_t + \frac{1}{\rho}\left\{\hat{R}^*_t - \left[\mathbb{E}_t\pi^*_{F,t+1} - (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}^*_{t+1} - \hat{Z}^*_t\right\}$$

- Market clearing:

$$\hat{Y}_{H,t} = c_y\left\{\nu\hat{C}_t + (1-\nu)\hat{C}^*_t + 2\theta\nu(1-\nu)\hat{T}_t\right\} + (1-c_y)\hat{G}_t$$
$$\hat{Y}_{F,t} = c_y\left\{(1-\nu)\hat{C}_t + \nu\hat{C}^*_t - 2\theta\nu(1-\nu)\hat{T}_t\right\} + (1-c_y)\hat{G}^*_t$$

- Aggregate supply:

$$\begin{aligned}\pi_{H,t} = \beta\mathbb{E}_t\pi_{H,t+1} + \kappa_H\Big\{&(\frac{\rho}{c_y}+\eta)\tilde{Y}_{H,t}\\ &- 2\nu(1-\nu)(\rho\theta-1)\tilde{T}_t + (1-\nu)\hat{f}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t\Big\}\\ \pi^*_{F,t} = \beta\mathbb{E}_t\pi^*_{F,t+1} + \kappa_F\Big\{&(\frac{\rho}{c_y}+\eta)\tilde{Y}_{F,t}\\ &+ 2\nu(1-\nu)(\rho\theta-1)\tilde{T}_t - (1-\nu)\hat{f}_t - \rho\frac{1-c_y}{c_y}\tilde{G}^*_t\Big\}\end{aligned}$$

- Risk sharing and demand imbalances:

$$\hat{Q}_t = \rho(\hat{C}_t - \hat{C}_t^*) + (\hat{Z}_t^* - \hat{Z}_t)$$
$$\hat{Q}_t = (2\nu - 1)\hat{T}_t$$

- First best allocation

$$(\rho + c_y\eta)\hat{Y}_{H,t}^{fb} = c_y 2\nu(1-\nu)(\rho\theta - 1)\hat{T}_t^{fb} + (1-c_y)\rho\hat{G}_t^{fb} + c_y(\nu\hat{Z}_t + (1-\nu)\hat{Z}_t^*)$$
$$(\rho + c_y\eta)\hat{Y}_{F,t}^{fb} = -c_y 2\nu(1-\nu)(\rho\theta - 1)\hat{T}_t^{fb} + (1-c_y)\rho\hat{G}_t^{*,fb} + c_y((1-\nu)\hat{Z}_t + \nu\hat{Z}_t^*)$$
$$c_y[4\nu(1-\nu)\rho\theta + (2\nu-1)^2]\hat{T}_t^{fb} = \rho(\hat{Y}_{H,t}^{fb} - \hat{Y}_{F,t}^{fb}) - (1-c_y)\rho(\hat{G}_t^{fb} - \hat{G}_t^{*,fb}) - c_y(2\nu-1)(\hat{Z}_t - \hat{Z}_t^*)$$
$$\eta\hat{Y}_{H,t}^{fb} = -\rho\hat{G}_t^{fb}$$
$$\eta\hat{Y}_{F,t}^{fb} = -\rho\hat{G}_t^{*,fb}$$

B.2.2 Equilibrium conditions under LCP

Given the path of the four policy instruments $\{R_t, R_t^*, G_t, G_t^*\}$, the equilibrium under flexible exchange rates consists of endogenous variables $\{\hat{Y}_{H,t}, \hat{Y}_{F,t}, \pi_{H,t}, \pi_{H,t}^*, \pi_{F,t}^*, \pi_{F,t}, \hat{C}_{H,t}, \hat{C}_{F,t}, \hat{T}_t, \hat{T}_t^*, \hat{Q}_t, \hat{m}_t\}$ and exogenous variables $\{\hat{Z}_t, \hat{Z}_t^*\}$ that satisfy:

- Aggregate demand:

$$\mathbb{E}_t\hat{C}_{t+1} = \hat{C}_t + \frac{1}{\rho}\left\{\hat{R}_t - \left[\mathbb{E}_t\pi_{H,t+1} + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}_{t+1} - \hat{Z}_t\right\}$$
$$\mathbb{E}_t\hat{C}_{t+1}^* = \hat{C}_t^* + \frac{1}{\rho}\left\{\hat{R}_t^* - \left[\mathbb{E}_t\pi_{F,t+1}^* + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1}^* - \hat{T}_t^*)\right] + \mathbb{E}_t\hat{Z}_{F,t+1} - \hat{Z}_t^*\right\}$$

- Market clearing:

$$\hat{Y}_{H,t} = c_y\left\{\nu\hat{C}_t + (1-\nu)\hat{C}_t^* + \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*)\right\} + (1-c_y)\hat{G}_t$$
$$\hat{Y}_{F,t} = c_y\left\{(1-\nu)\hat{C}_t + \nu\hat{C}_t^* - \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*)\right\} + (1-c_y)\hat{G}_t^*$$

- Aggregate supply:

$$
\begin{aligned}
\pi_{H,t} = \beta\mathbb{E}_t\pi_{H,t+1} + \kappa_H \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \\
- \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) + \frac{1-\nu}{2}(\hat{T}_t + \hat{T}_t^*) + (1-\nu)\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \Big\} \\
\pi_{H,t}^* = \beta\mathbb{E}_t\pi_{H,t+1}^* + \kappa_H \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \\
- \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - \frac{\nu}{2}(\hat{T}_t + \hat{T}_t^*) - \nu\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \Big\} \\
\pi_{F,t}^* = \beta\mathbb{E}_t\pi_{F,t+1}^* + \kappa_F \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t} \\
+ \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) + \frac{1-\nu}{2}(\hat{T}_t + \hat{T}_t^*) - (1-\nu)\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t^* \Big\} \\
\pi_{F,t} = \beta\mathbb{E}_t\pi_{F,t+1} + \kappa_F \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t} \\
+ \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - \frac{\nu}{2}(\hat{T}_t + \hat{T}_t^*) + \nu\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t^* \Big\}
\end{aligned}
$$

- Risk sharing:

$$
\begin{aligned}
\hat{Q}_t =& \rho(\hat{C}_t - \hat{C}_t^*) + (\hat{Z}_t^* - \hat{Z}_t) \\
\hat{Q}_t =& \hat{m}_t + \frac{2\nu - 1}{2}(\hat{T}_t - \hat{T}_t^*)
\end{aligned}
$$

- Dynamic equations for relative price:

$$
\begin{aligned}
\hat{T}_t - \hat{T}_{t-1} = \pi_{F,t} - \pi_{H,t} \\
\hat{T}_t^* - \hat{T}_{t-1}^* = \pi_{H,t}^* - \pi_{F,t}^*
\end{aligned}
$$

B.2.3 Equilibrium conditions under DCP

Given the path of the four policy instruments $\{R_t, R_t^*, G_t, G_t^*\}$, the equilibrium under flexible exchange rates consists of endogenous variables $\{\hat{Y}_{H,t}, \hat{Y}_{F,t}, \pi_{H,t}, \pi_{H,t}^*, \pi_{F,t}^*, \hat{C}_{H,t}, \hat{C}_{F,t}, \hat{T}_t, \hat{T}_t^*, \hat{Q}_t, \hat{m}_t\}$ and exogenous variables $\{\hat{Z}_t, \hat{Z}_t^*\}$ that satisfy:

- Aggregate demand:

$$
\begin{aligned}
\mathbb{E}_t\hat{C}_{t+1} = \hat{C}_t + \frac{1}{\rho}\Big\{ \hat{R}_t - \Big[\mathbb{E}_t\pi_{H,t+1} + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t) \Big] + \mathbb{E}_t\hat{Z}_{t+1} - \hat{Z}_t \Big\} \\
\mathbb{E}_t\hat{C}_{t+1}^* = \hat{C}_t^* + \frac{1}{\rho}\Big\{ \hat{R}_t^* - \Big[\mathbb{E}_t\pi_{F,t+1}^* + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1}^* - \hat{T}_t^*) \Big] + \mathbb{E}_t\hat{Z}_{F,t+1} - \hat{Z}_t^* \Big\}
\end{aligned}
$$

- Market clearing:

$$\hat{Y}_{H,t} = c_y \left\{ \nu \hat{C}_t + (1-\nu)\hat{C}_t^* + \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*) \right\} + (1-c_y)\hat{G}_t$$
$$\hat{Y}_{F,t} = c_y \left\{ (1-\nu) \hat{C}_t + \nu\hat{C}_t^* - \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*) \right\} + (1-c_y)\hat{G}_t^*$$

- Aggregate supply:

$$\begin{aligned}
\pi_{H,t} &= \beta \mathbb{E}_t \pi_{H,t+1} + \kappa_H \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \\
&\quad - \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) + 2(1-\nu)\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \Big\} \\
\pi_{H,t}^* &= \beta \mathbb{E}_t \pi_{H,t+1}^* + \kappa_H \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \\
&\quad - \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - 2\nu\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \Big\} \\
\pi_{F,t}^* &= \beta \mathbb{E}_t \pi_{F,t+1}^* + \kappa_F \Big\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t} \\
&\quad + \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - \rho\frac{1-c_y}{c_y}\tilde{G}_t^* \Big\}
\end{aligned}$$

- Risk sharing:

$$\hat{Q}_t = \rho(\hat{C}_t - \hat{C}_t^*) + (\hat{Z}_t^* - \hat{Z}_t)$$
$$\hat{Q}_t = \hat{m}_t + \frac{2\nu - 1}{2}(\hat{T}_t - \hat{T}_t^*)$$

- Dynamic equations for relative price:

$$\hat{T}_t^* - \hat{T}_{t-1}^* = \pi_{H,t}^* - \pi_{F,t}^*$$
$$\hat{m}_t = \frac{\hat{T}_t + \hat{T}_t^*}{2}$$

B.3 Welfare Loss Functions

B.3.1 Welfare Loss Function (PCP)

$$\begin{aligned}
2\mathcal{L}_t^W = &- \left(\frac{\rho}{c_y} + \eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D} + \eta\right)\left(\tilde{Y}_t^R\right)^2 - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D} + 1\right)\left(\tilde{G}_t^R\right)^2 \\
&+ 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R - \frac{\sigma}{2}\left(\frac{1}{\kappa_H}\pi_{H,t}^2 + \frac{1}{\kappa_F}{\pi^*}_{F,t}^{\,2}\right)
\end{aligned}$$

where $D \equiv (2\nu - 1)^2 + 4\rho\theta\nu(1-\nu)$

B.3.2 Welfare Loss Function (LCP)

$$
\begin{aligned}
2\mathcal{L}_t^W = & -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2-\left(\frac{\rho}{c_y D}+\eta\right)\left(\tilde{Y}_t^R\right)^2-\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2-\rho(1-c_y)\left(\frac{1-c_y}{c_y D}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& +2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W+2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R-\frac{c_y\theta\nu(1-\nu)}{D}\left(\hat{m}_t\right)^2-c_y\theta\nu(1-\nu)\left(\hat{T}_t^W\right)^2 \\
& -\frac{c_y\sigma}{2}\left(\frac{1-(1-\nu)c_y}{c_y\kappa_H}\pi_{H,t}^2+\frac{1-\nu}{\kappa_H}(\pi_{H,t}^*)^2+\frac{1-(1-\nu)c_y}{c_y\kappa_F}(\pi_{F,t}^*)^2+\frac{1-\nu}{\kappa_F}\pi_{F,t}^2\right)
\end{aligned}
$$

B.3.3 Welfare Loss Function (DCP)

$$
\begin{aligned}
2\mathcal{L}_t^W = & -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2-\left(\frac{\rho}{c_y D}+\eta\right)\left(\tilde{Y}_t^R\right)^2-\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2-\rho(1-c_y)\left(\frac{1-c_y}{c_y D}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& +2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W+2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R-\frac{c_y\theta\nu(1-\nu)}{D}\left(\hat{m}_t\right)^2-c_y\theta\nu(1-\nu)\left(\hat{T}_t^W\right)^2 \\
& -\frac{c_y\sigma}{2}\left(\frac{1-(1-\nu)c_y}{c_y\kappa_H}\pi_{H,t}^2+\frac{1-\nu}{\kappa_H}(\pi_{H,t}^*)^2+\frac{1}{c_y\kappa_F}(\pi_{F,t}^*)^2\right)
\end{aligned}
$$

B.4 First Best Allocation

The first-best allocation is the equilibrium in which prices are fully flexible, capital markets are complete, and the steady state markups are always neutralized with an appropriate subsidy. Since this efficient allocation is invariant to the pricing regime, it provides a useful benchmark for comparing the welfare implications across the three regimes. Any variable marked with a hat and the subscript fb (e.g., $\hat{X}_t^{fb}$) represents its log–linearized version under the first–best allocation.

The log-linearized risk-sharing condition under perfect risk-sharing is:

$$\hat{Q}_t^{fb}=\rho\left(\hat{C}_t^{fb}-\hat{C}_t^{*,fb}\right)-\left(\hat{Z}_t-\hat{Z}_t^*\right). \tag{B1}$$

Furthermore, log-linearizing the real exchange rate yields:

$$\hat{Q}_t^{fb}=(2\nu-1)\hat{T}_t^{fb}. \tag{B2}$$

Combining the preceding results with the market–clearing conditions yields:

$$\frac{\rho}{c_y}\hat{Y}_{H,t}^{fb}=\rho\hat{C}_t^{fb}+(1-\nu)[2\nu(\rho\theta-1)+1]\hat{T}_t^{fb}+\rho\frac{1-c_y}{c_y}\hat{G}_t^{fb}-(1-\nu)\left(\hat{Z}_t-\hat{Z}_t^*\right). \tag{B3}$$

Log-linearizing the Home optimal pricing equation under flexible prices yields:

$$\eta\hat{Y}_{H,t}^{fb}=-(1-\nu)\hat{T}_t^{fb}-\rho\hat{C}_t^{fb}+\hat{Z}_t. \tag{B4}$$

Eliminating $\hat{C}_t^{fb}$ by combining the equations (B3) and (B4) results in:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_{H,t}^{fb}=2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb}+\rho\frac{1-c_y}{c_y}\hat{G}_t^{fb}-(1-\nu)\left(\hat{Z}_t-\hat{Z}_t^{*}\right)+\hat{Z}_t. \tag{B5}$$

An analogous derivation for Foreign yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_{F,t}^{fb}=-2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb}+\rho\frac{1-c_y}{c_y}\hat{G}_t^{*,fb}+(1-\nu)\left(\hat{Z}_t-\hat{Z}_t^{*}\right)+\hat{Z}_t^{*}. \tag{B6}$$

Adding the equations (B5) and (B6) yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_t^{W,fb}=\rho\frac{1-c_y}{c_y}\hat{G}_t^{W,fb}+\hat{Z}_t^{W}. \tag{B7}$$

Subtracting the equation (B6) from (B5) yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_t^{R,fb}=2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb}+\rho\frac{1-c_y}{c_y}\hat{G}_t^{R,fb}+(2\nu-1)\hat{Z}_t^{R}. \tag{B8}$$

The efficient terms of trade is given by:

$$D\hat{T}_t^{fb}=\frac{2\rho}{c_y}\hat{Y}_t^{R,fb}-2\rho\frac{1-c_y}{c_y}\hat{G}_t^{R,fb}-2(2\nu-1)\hat{Z}_t^{R}, \tag{B9}$$

which is obtained by combining the two log-linearized aggregate market-clearing conditions with the risk-sharing condition (B1) and Equation (B2).

The first-best government spending follows:

$$\begin{aligned} & V'(N_t^{fb})=J'(G_t^{fb}). \\ \Leftrightarrow\quad & \eta\hat{Y}_{H,t}^{fb}=-\rho\hat{G}_t^{fb} \end{aligned} \tag{B10}$$

Finally, substituting (B8), (B9), and (B10) yields the relationship:

$$\hat{T}_t^{fb}=-2\eta\hat{Y}_t^{R} \tag{B11}$$

B.5 Derivation of the Quadratic Loss Function

In this section, we derive the quadratic welfare loss functions for each pricing regime, following closely the approach in Engel (2011). For all regimes, the aggregate period utility for Home and Foreign is given by:

$$\begin{aligned} & U(C_t)-V(N_t)+J(G_t)+U(C_t^{*})-V(N_t^{*})+J(G_t^{*}) \\ &\quad=\left(Z_t\frac{C_t^{1-\rho}}{1-\rho}-\frac{N_t^{1+\eta}}{1+\eta}+\Psi\frac{G_t^{1-\rho}}{1-\rho}\right)+\left(Z_t^{*}\frac{(C_t^{*})^{1-\rho}}{1-\rho}-\frac{(N_t^{*})^{1+\eta}}{1+\eta}+\Psi\frac{(G_t^{*})^{1-\rho}}{1-\rho}\right). \end{aligned} \tag{B12}$$

We assume an efficient steady state so that the subsidy satisfies $\frac{(\sigma-1)(1-\tau)}{\sigma}=1$. Moreover, in the efficient steady-state, $U'(C^{fb})=V'(N^{fb})=J'(G^{fb})$ and $U'(C^{*,fb})=V'(N^{*,fb})=J'(G^{*,fb})$ holds. The second-order approximation of the world utility (B12) yields:

$$
\begin{aligned}
2\mathcal{L}_t =& \hat{C}_t+\hat{C}_t^*-\frac{1}{c_y}\hat{Y}_{H,t}-\frac{1}{c_y}\hat{Y}_{F,t}+\frac{1-c_y}{c_y}\hat{G}_t+\frac{1-c_y}{c_y}\hat{G}_t^*+\hat{Z}_t\hat{C}_t+\hat{Z}_t^*\hat{C}_t^* \\
&+\frac{1-\rho}{2}\hat{C}_t^2+\frac{1-\rho}{2}(\hat{C}_t^*)^2-\frac{1}{c_y}\frac{1+\eta}{2}\hat{Y}_{H,t}^2-\frac{1}{c_y}\frac{1+\eta}{2}\hat{Y}_{F,t}^2+\frac{1-c_y}{c_y}\frac{1-\rho}{2}\hat{G}_t^2+\frac{1-c_y}{c_y}\frac{1-\rho}{2}(\hat{G}_t^*)^2+t.i.p.
\end{aligned}
\tag{B13}
$$

Subtracting the first–best welfare from (B13) and regrouping terms into the world and relative components gives:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right) =& \tilde{C}_t+\tilde{C}_t^*-\frac{1}{c_y}\tilde{Y}_{H,t}-\frac{1}{c_y}\tilde{Y}_{F,t}+\frac{1-c_y}{c_y}\tilde{G}_t+\frac{1-c_y}{c_y}\tilde{G}_t^*+\hat{Z}_t\tilde{C}_t+\hat{Z}_t^*\tilde{C}_t^* \\
&+\frac{1-\rho}{2}\tilde{C}_t^2+\frac{1-\rho}{2}(\tilde{C}_t^*)^2-\frac{1}{c_y}\frac{1+\eta}{2}\tilde{Y}_{H,t}^2-\frac{1}{c_y}\frac{1+\eta}{2}\tilde{Y}_{F,t}^2 \\
&+\frac{1-c_y}{c_y}\frac{1-\rho}{2}\tilde{G}_t^2+\frac{1-c_y}{c_y}\frac{1-\rho}{2}(\tilde{G}_t^*)^2 \\
&+(1-\rho)\left(\hat{C}_t^{fb}\tilde{C}_t+\hat{C}_t^{*,fb}\tilde{C}_t^*\right)-\frac{1}{c_y}(1+\eta)\left(\hat{Y}_{H,t}^{fb}\tilde{Y}_{H,t}+\hat{Y}_{F,t}^{fb}\tilde{Y}_{F,t}\right) \\
&+\frac{1-c_y}{c_y}(1-\rho)\left(\hat{G}_t^{fb}\tilde{G}_t+\hat{G}_t^{*,fb}\tilde{G}_t^*\right)+t.i.p.
\end{aligned}
$$

$$
\begin{aligned}
=& 2\tilde{C}_t^W-2\frac{1}{c_y}\tilde{Y}_t^W+2\frac{1-c_y}{c_y}\tilde{G}_t^W+2\hat{Z}_t^W\tilde{C}_t^W+2\hat{Z}_t^R\tilde{C}_t^R \\
&+(1-\rho)\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}(1+\eta)\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right] \\
&+\frac{1-c_y}{c_y}(1-\rho)\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right] \\
&+2(1-\rho)\left(\hat{C}_t^{W,fb}\tilde{C}_t^W+\hat{C}_t^{R,fb}\tilde{C}_t^R\right)-\frac{2}{c_y}(1+\eta)\left(\hat{Y}_t^{W,fb}\tilde{Y}_t^W+\hat{Y}_t^{R,fb}\tilde{Y}_t^R\right) \\
&+2\frac{1-c_y}{c_y}(1-\rho)\left(\hat{G}_t^{W,fb}\tilde{G}_t^W+\hat{G}_t^{R,fb}\tilde{G}_t^R\right)+t.i.p.
\end{aligned}
\tag{B14}
$$

The subsequent steps depend on the pricing regime; accordingly, we develop the quadratic loss function separately for each regime in the following subsections. Section B.5.1 and B.5.2 describe the steps to derive the welfare cost function under PCP. Section B.5.3 and B.5.4 illustrate the steps under LCP. Lastly, sections B.5.5 and B.5.6 show the steps under DCP. For each regime, we first state the relevant first– and second–order conditions and then develop the equation (B14) for that regime.

B.5.1 Useful First- and Second-Order Conditions (PCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu - 1)\hat{T}_t$), we obtain:

$$\hat{Y}_{H,t} = c_y \left[\hat{C}_t + \frac{D - (2\nu - 1)}{2\rho} \hat{T}_t - \frac{2(1-\nu)}{\rho} \hat{Z}_t^R \right] + (1 - c_y)\hat{G}_t, \tag{B15}$$

$$\hat{Y}_{H,t} = c_y \left[\hat{C}_t^* + \frac{D + (2\nu - 1)}{2\rho} \hat{T}_t + \frac{2\nu}{\rho} \hat{Z}_t^R \right] + (1 - c_y)\hat{G}_t, \tag{B16}$$

$$\hat{Y}_{F,t} = c_y \left[\hat{C}_t^* - \frac{D - (2\nu - 1)}{2\rho} \hat{T}_t + \frac{2(1-\nu)}{\rho} \hat{Z}_t^R \right] + (1 - c_y)\hat{G}_t^*. \tag{B17}$$

Then, by combining the equations (B15), (B16), and (B17), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, and the shock as follow:

$$\hat{C}_t^W = \frac{1}{c_y}\hat{Y}_t^W - \frac{1-c_y}{c_y}\hat{G}_t^W, \tag{B18}$$

$$\hat{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^R + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \tag{B19}$$

$$\hat{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^R - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B20}$$

Adding (B15) to (B17) yields (B18), subtracting (B17) from (B16) yields (B20), and substituting (B20) into the risk-sharing condition yields (B19).

Similarly, the corresponding equations under efficient equilibrium are:

$$\hat{C}_t^{W,fb} = \frac{1}{c_y}\hat{Y}_t^{W,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{W,fb}, \tag{B21}$$

$$\hat{C}_t^{R,fb} = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^{R,fb} + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^{R,fb}, \tag{B22}$$

$$\hat{T}_t^{fb} = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^{R,fb} - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B23}$$

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W, \tag{B24}$$

$$\tilde{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\tilde{G}_t^R, \tag{B25}$$

$$\tilde{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R. \tag{B26}$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \Delta_{H,t} \left\{ \left(\frac{P_{H,t}}{P_t} \right)^{-\theta} \left[\nu C_t + (1-\nu) Q^\theta C_t^* \right] + G_t \right\},$$
$$Y_{F,t} = \Delta_{F,t}^* \left\{ \left(\frac{P_{F,t}^*}{P_t^*} \right)^{-\theta} \left[(1-\nu) C_t + \nu Q^\theta C_t^* \right] + G_t^* \right\},$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\sigma} di$ and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*} \right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following[1]:

$$\begin{aligned} 2\hat{C}_t^W - \frac{2}{c_y}\hat{Y}_t^W + \frac{2(1-c_y)}{c_y}\hat{G}_t^W =& - \left[\left(\hat{C}_t^W\right)^2 + \left(\hat{C}_t^R\right)^2 \right] + \frac{1}{c_y} \left[\left(\hat{Y}_t^W\right)^2 + \left(\hat{Y}_t^R\right)^2 \right] \\ & - \frac{1-c_y}{c_y} \left[\left(\hat{G}_t^W\right)^2 + \left(\hat{G}_t^R\right)^2 \right] - \theta\nu(1-\nu)\hat{T}_t^2 \\ & - \frac{1}{c_y}\frac{\sigma}{2\kappa} \left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2 \right] \end{aligned} \tag{B27}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (B27), we get:

$$\begin{aligned} 2\tilde{C}_t^W - \frac{2}{c_y}\tilde{Y}_t^W + \frac{2(1-c_y)}{c_y}\tilde{G}_t^W =& - \left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2 \right] + \frac{1}{c_y} \left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2 \right] \\ & - \frac{1-c_y}{c_y} \left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2 \right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\ & - \left(\hat{C}_t^{fb,W}\tilde{C}_t^W + \hat{C}_t^{fb,R}\tilde{C}_t^R \right) + \frac{2}{c_y} \left(\hat{Y}_t^{fb,W}\tilde{Y}_t^W + \hat{Y}_t^{fb,R}\tilde{Y}_t^R \right) \\ & - \frac{2(1-c_y)}{c_y} \left(\hat{G}_t^{fb,W}\tilde{G}_t^W + \hat{G}_t^{fb,R}\tilde{G}_t^R \right) - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\ & - \frac{1}{c_y}\frac{\sigma}{2\kappa} \left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2 \right]. \end{aligned} \tag{B28}$$

Equations (B21), (B22), (B23), (B24), (B25), (B26), and (B28) will be used in the next subsection.

B.5.2 Further Manipulation of the Welfare Loss Function (PCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (B28) into (B14), we

[1]The inflation rates appear because of the second-order approximation of price dispersion terms. Please refer to Woodford (2003) chapter 6 for more detail.

obtain:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-2\rho\hat{C}_t^{R,fb}\tilde{C}_t^R+2\hat{Z}_t^R\tilde{C}_t^R-2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{R,fb}\tilde{G}_t^R\\
&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\tilde{T}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2+\left(\hat{\pi}_{F,t}^*\right)^2\right]+t.i.p.
\end{aligned}
\tag{B29}
$$

To recast the welfare loss in terms of output gaps, fiscal gaps, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R=\frac{2\nu-1}{2\rho}\tilde{T}_t$[2] and the equation (B10), we can rearrange the first two rows of the equation (B29) as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-\left[(2\nu-1)\hat{C}_t^{R,fb}+2\theta\nu(1-\nu)\hat{T}_t^{fb}-\frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\tilde{T}_t\\
&-2\eta\hat{Y}_t^{R,fb}\left(\frac{1}{c_y}\tilde{Y}_t^R-\frac{1-c_y}{c_y}\tilde{G}_t^R\right)\\
&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\tilde{T}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2+\left(\hat{\pi}_{F,t}^*\right)^2\right]+t.i.p.
\end{aligned}
$$

[2] This relationship can be easily derived from the equations (B25) and (B26).

Then, substituting the equations (B22) and (B23) into the first row, and the equation (B26) into the second row yields:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& -\left(\frac{1}{c_y}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right)\tilde{T}_t \\
& - \frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\tilde{T}_t \\
& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Using the equation (B23) and the relation between the flexible relative price and the flexible relative output gap (B11) rids out the first two rows.

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (B7), (B21) and (B24) to yield:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& 2\eta\hat{Y}_t^{W,fb}\left(\frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W\right) \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Then, using the equation (B10) will substitute out the interaction terms:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (B24), (B25), and (B26), and some rearrangement, we can derive the welfare loss function as follows[3]:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\left(\frac{\rho}{c_y} + \eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D} + \eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D} + 1\right)\left(\tilde{G}_t^R\right)^2 \\
& + 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R \\
& - \frac{\sigma}{\kappa}\left[(\hat{\pi}_t^W)^2 + (\hat{\pi}_{ppi,t}^R)^2\right].
\end{aligned} \tag{B30}
$$

B.5.3 Useful First- and Second-Order Conditions (LCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu - 1)\hat{T}_t + \hat{m}_t$), we obtain:

$$
\hat{Y}_{H,t} = c_y\left\{\hat{C}_t + \frac{D-(2\nu-1)}{2\rho}\hat{T}_t - \frac{1-\nu}{\rho}\hat{m}_t - \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t, \tag{B31}
$$

$$
\hat{Y}_{H,t} = c_y\left\{\hat{C}_t^* + \frac{D+(2\nu-1)}{2\rho}\hat{T}_t + \frac{1-\nu}{\rho}\hat{m}_t + \frac{2\nu}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t, \tag{B32}
$$

$$
\hat{Y}_{F,t} = c_y\left\{\hat{C}_t^* - \frac{D-(2\nu-1)}{2\rho}\hat{T}_t + \frac{\nu}{\rho}\hat{m}_t + \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t^*. \tag{B33}
$$

Then, by combining the equations (B31), (B32), and (B33), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, currency misalignment, and the shock as follow:

$$
\hat{C}_t^W = \frac{1}{c_y}\hat{Y}_t^W - \frac{1-c_y}{c_y}\hat{G}_t^W, \tag{B34}
$$

$$
\hat{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{m}_t + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \tag{B35}
$$

$$
\hat{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B36}
$$

[3] We normalize the function by c_y.

Adding (B31) to (B33) yields (B34), subtracting (B33) from (B32) yields (B36), and substituting (B36) into the risk-sharing condition yields (B35).

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W, \tag{B37}$$

$$\tilde{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\tilde{G}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{m}_t, \tag{B38}$$

$$\tilde{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t. \tag{B39}$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \nu\Delta_{H,t}\left(\frac{P_{H,t}}{P_t}\right)^{-\theta} C_t + (1-\nu)\Delta_{H,t}^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\theta} C_t^* + \Delta_{H,t}G_t,$$

$$Y_{F,t} = (1-\nu)\Delta_{F,t}\left(\frac{P_{F,t}}{P_t}\right)^{-\theta} C_t + \nu\Delta_{F,t}^*\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\theta} C_t^* + \Delta_{F,t}^*G_t^*,$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\sigma} di$, $\Delta_{H,t}^* \equiv \int_0^1 \left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\sigma} di$, $\Delta_{F,t} \equiv \int_0^1 \left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\sigma} di$ and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following:

$$\begin{aligned}2\hat{C}_t^W - \frac{2}{c_y}\hat{Y}_t^W + \frac{2(1-c_y)}{c_y}\hat{G}_t^W = &-\left[\left(\hat{C}_t^W\right)^2 + \left(\hat{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\hat{Y}_t^W\right)^2 + \left(\hat{Y}_t^R\right)^2\right] \\ &- \frac{1-c_y}{c_y}\left[\left(\hat{G}_t^W\right)^2 + \left(\hat{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\hat{T}_t^2 \\ &- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\ &\left.+[1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\}.\end{aligned} \tag{B40}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (B40), we get:

$$
\begin{aligned}
2\tilde{C}_t^W - \frac{2}{c_y}\tilde{Y}_t^W + \frac{2(1-c_y)}{c_y}\tilde{G}_t^W = & - \left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \left(\hat{C}_t^{fb,W}\tilde{C}_t^W + \hat{C}_t^{fb,R}\tilde{C}_t^R\right) + \frac{2}{c_y}\left(\hat{Y}_t^{fb,W}\tilde{Y}_t^W + \hat{Y}_t^{fb,R}\tilde{Y}_t^R\right) \\
& - \frac{2(1-c_y)}{c_y}\left(\hat{G}_t^{fb,W}\tilde{G}_t^W + \hat{G}_t^{fb,R}\tilde{G}_t^R\right) - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. +[1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\}. \qquad \text{(B41)}
\end{aligned}
$$

Equations (B21), (B22), (B23), (B37), (B38), (B39), and (B41) will be used in the next subsection.

B.5.4 Further Manipulation of the Welfare Loss Function (LCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (B41) into (B14), we obtain:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & - 2\rho\hat{C}_t^{R,fb}\tilde{C}_t^R + 2\hat{Z}_t^R\tilde{C}_t^R - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{R,fb}\tilde{G}_t^R \\
& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. +[1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p. \qquad \text{(B42)}
\end{aligned}
$$

To recast the welfare loss in terms of output gaps, fiscal gaps, relative price gaps, currency misalignment, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R = \frac{2\nu-1}{2\rho}\tilde{T}_t + \frac{1}{2\rho}\hat{m}_t$[4] and

[4]This relationship can be easily derived from the equations (B38) and (B39).

the equation (B10), we can rearrange the first two rows of the equation (B42) as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & - \left\{(2\nu-1)\hat{C}_t^{R,fb} + 2\theta\nu(1-\nu)\hat{T}_t^{fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t \\
& - \left\{\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right\}\hat{m}_t \\
& - 2\eta\hat{Y}_t^{R,fb}\left(\frac{1}{c_y}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\tilde{G}_t^R\right) \\
& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Then, substituting the equations (B22) and (B23) into the first row, and the equation (B39) into the third row yields:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & - \left\{\frac{1}{c_y}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t \\
& - \left\{\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right\}\hat{m}_t \\
& - \frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\left\{\tilde{T}_t + \frac{2\nu-1}{D}\hat{m}_t\right\} \\
& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Using the equation (B1), (B2), (B23) and the relation between the flexible relative price and the flexible relative output gap (B11) rids out the first three rows.

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (B7), (B21) and (B37) to yield:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & 2\eta\hat{Y}_t^{W,fb}\left(\frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W\right) \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Then, using the equation (B10) will substitute out the interaction terms:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (B37), (B38), and (B39), and some rearrangement, we can derive the welfare loss function as follows[5]:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)= & -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2-\left(\frac{\rho}{c_y D}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& -\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2-\rho(1-c_y)\left(\frac{1-c_y}{c_y D}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& +2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W+2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R \\
& -\frac{c_y\theta\nu(1-\nu)}{D}\hat{m}_t^2 \\
& -\frac{\sigma}{\kappa}\left\{\left(\hat{\pi}_t^W\right)^2+\left(\hat{\pi}_{cpi,t}^R\right)^2-2\left(1-c_y\right)\left(1-v\right)\hat{\pi}_{cpi,t}^R\Delta\hat{T}_t\right. \\
& \left.+\left[c_y(2v-1)+(1-v)\right](1-v)\left(\Delta\hat{T}_t\right)^2\right\}.
\end{aligned} \tag{B43}
$$

B.5.5 Useful First- and Second-Order Conditions (DCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t=(2\nu-1)\hat{T}_t^R+\hat{m}_t$), we obtain:

$$\hat{Y}_{H,t}=c_y\left[\hat{C}_t+\frac{D-(2\nu-1)}{2\rho}\hat{T}_t-\frac{1-\nu}{\rho}\hat{m}_t-\frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right]+(1-c_y)\hat{G}_t, \tag{B44}$$

$$\hat{Y}_{H,t}=c_y\left[\hat{C}_t^*+\frac{D+(2\nu-1)}{2\rho}\hat{T}_t+\frac{1-\nu}{\rho}\hat{m}_t+\frac{2\nu}{\rho}\hat{Z}_t^R\right]+(1-c_y)\hat{G}_t, \tag{B45}$$

$$\hat{Y}_{F,t}=c_y\left[\hat{C}_t^*-\frac{D-(2\nu-1)}{2\rho}\hat{T}_t+\frac{\nu}{\rho}\hat{m}_t+\frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right]+(1-c_y)\hat{G}_t^*. \tag{B46}$$

Then, by combining the equations (B44), (B45), and (B46), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, currency misalignment, and the shock as follow:

$$\hat{C}_t^W=\frac{1}{c_y}\hat{Y}_t^W-\frac{1-c_y}{c_y}\hat{G}_t^W, \tag{B47}$$

$$\hat{C}_t^R=\frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^R-\frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^R+\frac{2\theta\nu(1-\nu)}{D}\hat{m}_t+\frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \tag{B48}$$

$$\hat{T}_t^R=\frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^R-\frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^R-\frac{2\nu-1}{D}\hat{m}_t-\frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B49}$$

Adding (B44) to (B46) yields (B47), subtracting (B46) from (B45) yields (B49), and substituting (B49) into the risk-sharing condition yields (B48).

[5] We normalize the function by c_y.

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W, \tag{B50}$$

$$\tilde{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\tilde{G}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{m}_t, \tag{B51}$$

$$\tilde{T}_t^R = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t. \tag{B52}$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \nu\Delta_{H,t}\left(\frac{P_{H,t}}{P_t}\right)^{-\theta} C_t + (1-\nu)\Delta_{H,t}^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\theta} C_t^* + \Delta_{H,t}G_t,$$

$$Y_{F,t} = \Delta_{F,t}^*\left\{\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\theta}\left[(1-\nu)C_t + \nu Q^\theta C_t^*\right] + G_t^*\right\},$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\sigma} di$, $\Delta_{H,t}^* \equiv \int_0^1 \left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\sigma} di$, and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following:

$$\begin{aligned}
2\hat{C}_t^W - \frac{2}{c_y}\hat{Y}_t^W + \frac{2(1-c_y)}{c_y}\hat{G}_t^W = &-\left[\left(\hat{C}_t^W\right)^2 + \left(\hat{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\hat{Y}_t^W\right)^2 + \left(\hat{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\left[\left(\hat{G}_t^W\right)^2 + \left(\hat{G}_t^R\right)^2\right] - \frac{1}{2}\theta\nu(1-\nu)\left[\left(\hat{T}_t\right)^2 + \left(\hat{T}_t^*\right)^2\right] \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2 + (\pi_{F,t}^*)^2\right\}.
\end{aligned} \tag{B53}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (B53) and rearranging the relative price terms, we get:

$$\begin{aligned}
2\tilde{C}_t^W - \frac{2}{c_y}\tilde{Y}_t^W + \frac{2(1-c_y)}{c_y}\tilde{G}_t^W = &-\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\left[\left(\hat{T}_t^R\right)^2 + \hat{m}_t^2\right] \\
&- \left(\hat{C}_t^{W,fb}\tilde{C}_t^W + \hat{C}_t^{R,fb}\tilde{C}_t^R\right) + \frac{2}{c_y}\left(\hat{Y}_t^{W,fb}\tilde{Y}_t^W + \hat{Y}_t^{R,fb}\tilde{Y}_t^R\right) \\
&- \frac{2(1-c_y)}{c_y}\left(\hat{G}_t^{W,fb}\tilde{G}_t^W + \hat{G}_t^{R,fb}\tilde{G}_t^R\right) - 2\theta\nu(1-\nu)\hat{T}^{R,fb}\tilde{T}_t^R \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2 + (\pi_{F,t}^*)^2\right\}.
\end{aligned} \tag{B54}$$

Equations (B21), (B22), (B23), (B50), (B51), (B52), and (B54) will be used in the next subsection.

B.5.6 Further Manipulation of the Welfare Loss Function (DCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (B54) into (B14), we obtain:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- 2\rho \hat{C}_t^{R,fb}\tilde{C}_t^R + 2\hat{Z}_t^R\tilde{C}_t^R - 2\theta\nu(1-\nu)\hat{T}^{R,fb}\tilde{T}_t^R \\
&- \frac{2}{c_y}\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{R,fb}\tilde{G}_t^R \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2 - \theta\nu(1-\nu)\hat{m}_t^2 \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2 + (\pi_{F,t}^*)^2\right\} + t.i.p.
\end{aligned}
\tag{B55}
$$

To recast the welfare loss in terms of output gaps, fiscal gaps, relative price gaps, currency misalignment, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R = \frac{2\nu-1}{2\rho}\tilde{T}_t^R + \frac{1}{2\rho}\hat{m}_t$[6] and the equation (B10), we can rearrange the first two rows of the equation (B55) as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- \left[(2\nu-1)\hat{C}_t^{R,fb} + 2\theta\nu(1-\nu)\hat{T}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\tilde{T}_t^R \\
&- \left(\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right)\hat{m}_t \\
&- 2\eta\hat{Y}_t^{R,fb}\left(\frac{1}{c_y}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\tilde{G}_t^R\right) \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2 - \theta\nu(1-\nu)\hat{m}_t^2 \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2 + (\pi_{F,t}^*)^2\right\} + t.i.p.
\end{aligned}
$$

[6] This relationship can be easily derived from the equations (B51) and (B52).

Then, substituting the equations (B22) and (B23) into the first row, and the equation (B52) into the third row yields:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-\left[\frac{1}{c_y}\hat{Y}_t^{R,fb}-\frac{1-c_y}{c_y}\hat{G}_t^{R,fb}-\frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\tilde{T}_t^R\\
&-\left(\hat{C}_t^{R,fb}-\frac{1}{\rho}\hat{Z}_t^R\right)\hat{m}_t\\
&-\frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\left(\tilde{T}_t^R+\frac{2\nu-1}{D}\hat{m}_t\right)\\
&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2-\theta\nu(1-\nu)\hat{m}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2+c_y(1-\nu)(\pi_{H,t}^*)^2+(\pi_{F,t}^*)^2\right\}+t.i.p.
\end{aligned}
$$

Using the equation (B1), (B2), (B23) and the relation between the flexible relative price and the flexible relative output gap (B11) rids out the first three rows.

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2-\theta\nu(1-\nu)\hat{m}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2+c_y(1-\nu)(\pi_{H,t}^*)^2+(\pi_{F,t}^*)^2\right\}+t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (B7), (B21) and (B50) to yield:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&2\eta\hat{Y}_t^{W,fb}\left(\frac{1}{c_y}\tilde{Y}_t^W-\frac{1-c_y}{c_y}\tilde{G}_t^W\right)\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2-\theta\nu(1-\nu)\hat{m}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2+c_y(1-\nu)(\pi_{H,t}^*)^2+(\pi_{F,t}^*)^2\right\}+t.i.p.
\end{aligned}
$$

Then, using the equation (B10) will substitute out the interaction terms:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\left(\tilde{T}_t^R\right)^2 - \theta\nu(1-\nu)\hat{m}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2 + (\pi_{F,t}^*)^2\right\} + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t^R$ using the equations (B50), (B51), and (B52), and some rearrangement, we can derive the welfare loss function as follows[7]:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\left(\frac{\rho}{c_y} + \eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D} + \eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D} + 1\right)\left(\tilde{G}_t^R\right)^2 \\
& + 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R \\
& - \frac{c_y\theta\nu(1-\nu)(D+1)}{D}\hat{m}_t^2 \\
& - \frac{c_y\sigma}{2}\left[\frac{1-(1-\nu)c_y}{c_y\kappa_H}\hat{\pi}_{H,t}^2 + \frac{1-\nu}{\kappa_H}(\hat{\pi}_{H,t}^*)^2 + \frac{1}{c_y\kappa_F}(\hat{\pi}_{F,t}^*)^2\right].
\end{aligned} \tag{B56}
$$

[7] We normalize the function by c_y.

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