Online Appendix for Currency Misalignment and Optimal Capital Controls

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In this online appendix, we derive the quadratic approximation of the loss function used for evaluating welfare and deriving the optimal policies.

1 First Best Allocation

The first-best allocation is the equilibrium in which prices are fully flexible, capital markets are complete, and the steady state markups are always neutralized with an appropriate subsidy. Since this efficient allocation is invariant to the pricing regime, it provides a useful benchmark for comparing the welfare implications across the three regimes. Any variable marked with a hat and the subscript fb (e.g., $\hat{X}_t^{fb}$) represents its log-linearized version under the first-best allocation.

The log-linearized risk-sharing condition under perfect risk-sharing is:

$$\hat{Q}_t^{fb} = \rho\left(\hat{C}_t^{fb} - \hat{C}_t^{*,fb}\right) - \left(\hat{Z}_t - \hat{Z}_t^*\right). \tag{1}$$

Furthermore, log-linearizing the real exchange rate yields:

$$\hat{Q}_t^{fb} = (2\nu - 1)\hat{T}_t^{fb}. \tag{2}$$

Combining the preceding results with the market-clearing conditions yields:

$$\rho\hat{Y}_{H,t}^{fb} = \rho\hat{C}_t^{fb} + (1-\nu)[2\nu(\rho\theta - 1) + 1]\hat{T}_t^{fb} - (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right). \tag{3}$$

Log-linearizing the Home optimal pricing equation under flexible prices yields:

$$\eta\hat{Y}_{H,t}^{fb} = -(1-\nu)\hat{T}_t^{fb} - \rho\hat{C}_t^{fb} + \hat{Z}_t + (1+\eta)\hat{A}_t. \tag{4}$$

Eliminating $\hat{C}_t^{fb}$ by combining the equations (3) and (4) results in:

$$(\rho+\eta)\,\hat{Y}_{H,t}^{fb} = 2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb} - (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right) + \hat{Z}_t + (1+\eta)\hat{A}_t. \tag{5}$$

An analogous derivation for Foreign yields:

$$(\rho+\eta)\,\hat{Y}_{F,t}^{fb} = -2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb} + (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right) + \hat{Z}_t^* + (1+\eta)\hat{A}_t^*. \tag{6}$$

Adding the equations (5) and (6) yields:

$$(\rho+\eta)\,\hat{Y}_t^{W,fb} = \hat{Z}_t^W + (1+\eta)\hat{A}_t^W. \tag{7}$$

Subtracting the equation (6) from (5) yields:

$$(\rho+\eta)\,\hat{Y}_t^{R,fb} = 2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb} + (2\nu-1)\hat{Z}_t^R + (1+\eta)\hat{A}_t^R. \tag{8}$$

The efficient terms of trade is given by:

$$D\hat{T}_t^{fb} = 2\rho\hat{Y}_t^{R,fb} - 2(2\nu-1)\hat{Z}_t^R, \tag{9}$$

which is obtained by combining the two log-linearized aggregate market-clearing conditions with the risk-sharing condition (1) and Equation (2).

Finally, substituting (8) and (9) yields the relationship:

$$\hat{T}_t^{fb} = -2\eta\hat{Y}_t^{R,fb} + (1+\eta)\hat{A}_t^R \tag{10}$$

2 Derivation of the Quadratic Loss Function

In this section, we derive the quadratic welfare loss functions for each pricing regime, following closely the approach in Engel (2011). For all regimes, the aggregate period utility for Home and Foreign is given by:

$$\begin{aligned} &U(C_t) - V(N_t) + U(C_t^*) - V(N_t^*) \\ &\quad = \left(Z_t\frac{C_t^{1-\rho}}{1-\rho} - \frac{N_t^{1+\eta}}{1+\eta}\right) + \left(Z_t^*\frac{(C_t^*)^{1-\rho}}{1-\rho} - \frac{(N_t^*)^{1+\eta}}{1+\eta}\right). \end{aligned} \tag{11}$$

We assume an efficient steady state so that the subsidy satisfies $\frac{(\sigma-1)(1-\tau)}{\sigma} = 1$. Moreover, in the efficient steady-state, $U'(C^{fb}) = V'(N^{fb})$ and $U'(C^{*,fb}) = V'(N^{*,fb})$ holds. The second-order approximation of the world utility (11) yields:

$$\begin{aligned} -2\mathcal{L}_t =& \hat{C}_t + \hat{C}_t^* - \hat{Y}_{H,t} - \hat{Y}_{F,t} + \hat{Z}_t\hat{C}_t + \hat{Z}_t^*\hat{C}_t^* \\ &+ \frac{1-\rho}{2}\hat{C}_t^2 + \frac{1-\rho}{2}(\hat{C}_t^*)^2 - \frac{1+\eta}{2}\hat{Y}_{H,t}^2 - \frac{1+\eta}{2}\hat{Y}_{F,t}^2 \\ &+ (1+\eta)\hat{A}_t\hat{Y}_{H,t} + (1+\eta)\hat{A}_t^*\hat{Y}_{F,t} + t.i.p. \end{aligned} \tag{12}$$

Adding the first-best welfare from (12) and regrouping terms into the world and relative components gives:

$$\begin{aligned} -2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& \tilde{C}_t + \tilde{C}_t^* - \tilde{Y}_{H,t} - \tilde{Y}_{F,t} + \hat{Z}_t\tilde{C}_t + \hat{Z}_t^*\tilde{C}_t^* + (1+\eta)\hat{A}_t\tilde{Y}_{H,t} + (1+\eta)\hat{A}_t^*\tilde{Y}_{F,t} \\ &+ \frac{1-\rho}{2}\tilde{C}_t^2 + \frac{1-\rho}{2}(\tilde{C}_t^*)^2 - \frac{1+\eta}{2}\tilde{Y}_{H,t}^2 - \frac{1+\eta}{2}\tilde{Y}_{F,t}^2 \\ &+ (1-\rho)\left(\hat{C}_t^{fb}\tilde{C}_t + \hat{C}_t^{*,fb}\tilde{C}_t^*\right) - (1+\eta)\left(\hat{Y}_{H,t}^{fb}\tilde{Y}_{H,t} + \hat{Y}_{F,t}^{fb}\tilde{Y}_{F,t}\right) + t.i.p. \\ \\ =& 2\tilde{C}_t^W - 2\tilde{Y}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W + 2\hat{Z}_t^R\tilde{C}_t^R + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\ &+ (1-\rho)\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - (1+\eta)\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\ &+ 2(1-\rho)\left(\hat{C}_t^{W,fb}\tilde{C}_t^W + \hat{C}_t^{R,fb}\tilde{C}_t^R\right) - 2(1+\eta)\left(\hat{Y}_t^{W,fb}\tilde{Y}_t^W + \hat{Y}_t^{R,fb}\tilde{Y}_t^R\right) + t.i.p. \end{aligned} \tag{13}$$

The subsequent steps depend on the pricing regime; accordingly, we develop the quadratic loss function separately for each regime in the following subsections. Section 2.1 and 2.2 describe the steps to derive the welfare cost function under PCP. Section 2.3 and 2.4 illustrate the steps under LCP. For each regime, we first state the relevant first and second-order conditions, then develop the equation (13) for that regime.

2.1 Useful First- and Second-Order Conditions (PCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu - 1)\hat{T}_t$), we obtain:

$$\hat{Y}_{H,t} = \hat{C}_t + \frac{D - (2\nu - 1)}{2\rho}\hat{T}_t - \frac{1-\nu}{\rho}\hat{F}_t - \frac{2(1-\nu)}{\rho}\hat{Z}_t^R, \tag{14}$$

$$\hat{Y}_{H,t} = \hat{C}_t^* + \frac{D + (2\nu - 1)}{2\rho}\hat{T}_t + \frac{\nu}{\rho}\hat{F}_t + \frac{2\nu}{\rho}\hat{Z}_t^R, \tag{15}$$

$$\hat{Y}_{F,t} = \hat{C}_t^* - \frac{D - (2\nu - 1)}{2\rho}\hat{T}_t + \frac{1-\nu}{\rho}\hat{F}_t + \frac{2(1-\nu)}{\rho}\hat{Z}_t^R. \tag{16}$$

Then, by combining the equations (14), (15), and (16), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, and the shock as follow:

$$\hat{C}_t^W = \hat{Y}_t^W, \tag{17}$$

$$\hat{C}_t^R = \frac{2\nu - 1}{D}\hat{Y}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{F}_t + \frac{D - (2\nu - 1)^2}{\rho D}\hat{Z}_t^R, \tag{18}$$

$$\hat{T}_t = \frac{2\rho}{D}\hat{Y}_t^R - \frac{2\nu - 1}{D}\hat{F}_t - \frac{2(2\nu - 1)}{D}\hat{Z}_t^R. \tag{19}$$

Adding (14) to (16) yields (17), subtracting (16) from (15) yields (19), and substituting (19) into the risk-sharing condition yields (18).

Similarly, the corresponding equations under efficient equilibrium are:

$$\hat{C}_t^{W,fb} = \hat{Y}_t^{W,fb}, \tag{20}$$

$$\hat{C}_t^{R,fb} = \frac{2\nu - 1}{D}\hat{Y}_t^{R,fb} + \frac{D - (2\nu - 1)^2}{\rho D}\hat{Z}_t^{R,fb}, \tag{21}$$

$$\hat{T}_t^{fb} = \frac{2\rho}{D}\hat{Y}_t^{R,fb} - \frac{2(2\nu - 1)}{D}\hat{Z}_t^R. \tag{22}$$

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \tilde{Y}_t^W, \tag{23}$$

$$\tilde{C}_t^R = \frac{2\nu - 1}{D}\tilde{Y}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{F}_t, \tag{24}$$

$$\tilde{T}_t = \frac{2\rho}{D}\tilde{Y}_t^R - \frac{2\nu - 1}{D}\hat{F}_t. \tag{25}$$

The resource constraint for each country can be written as follows:

$$Y_{H,t} = \Delta_{H,t} \left\{ \left(\frac{P_{H,t}}{P_t} \right)^{-\theta} \left[\nu C_t + (1-\nu) Q^{\theta} C_t^* \right] \right\},$$
$$Y_{F,t} = \Delta_{F,t}^* \left\{ \left(\frac{P_{F,t}^*}{P_t^*} \right)^{-\theta} \left[(1-\nu) C_t + \nu Q^{\theta} C_t^* \right] \right\},$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\sigma} di$ and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*} \right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following[1]:

$$\begin{aligned} 2\hat{C}_t^W - 2\hat{Y}_t^W = &- \left[\left(\hat{C}_t^W \right)^2 + \left(\hat{C}_t^R \right)^2 \right] + \left[\left(\hat{Y}_t^W \right)^2 + \left(\hat{Y}_t^R \right)^2 \right] \\ &- \theta\nu(1-\nu)\hat{T}_t^2 - \frac{\sigma}{2\kappa} \left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^* \right)^2 \right] \end{aligned} \tag{26}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (26), we get:

$$\begin{aligned} 2\tilde{C}_t^W - 2\tilde{Y}_t^W = &- \left[\left(\tilde{C}_t^W \right)^2 + \left(\tilde{C}_t^R \right)^2 \right] + \left[\left(\tilde{Y}_t^W \right)^2 + \left(\tilde{Y}_t^R \right)^2 \right] \\ &- \theta\nu(1-\nu)\tilde{T}_t^2 \\ &- \left(\hat{C}_t^{fb,W} \tilde{C}_t^W + \hat{C}_t^{fb,R} \tilde{C}_t^R \right) + 2 \left(\hat{Y}_t^{fb,W} \tilde{Y}_t^W + \hat{Y}_t^{fb,R} \tilde{Y}_t^R \right) \\ &- 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\ &- \frac{\sigma}{2\kappa} \left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^* \right)^2 \right]. \end{aligned} \tag{27}$$

Equations (20), (21), (22), (23), (24), (25), and (27) will be used in the next subsection.

2.2 Further Manipulation of the Welfare Loss Function (PCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (27) into (13), we obtain:

$$\begin{aligned} -2\left(\mathcal{L}_t - \mathcal{L}_t^{fb} \right) = &- 2\rho\hat{C}_t^{R,fb}\tilde{C}_t^R + 2\hat{Z}_t^R\tilde{C}_t^R - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\ &- 2\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\ &- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\ &- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W \\ &- \rho\left[\left(\tilde{C}_t^W \right)^2 + \left(\tilde{C}_t^R \right)^2 \right] - \eta\left[\left(\tilde{Y}_t^W \right)^2 + \left(\tilde{Y}_t^R \right)^2 \right] \\ &- \theta\nu(1-\nu)\tilde{T}_t^2 \\ &- \frac{\sigma}{2\kappa} \left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^* \right)^2 \right] + t.i.p. \end{aligned} \tag{28}$$

[1] The inflation rates appear because of the second-order approximation of price dispersion terms. Please refer to Woodford (2003) chapter 6 for more detail.

To recast the welfare loss in terms of output gaps and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R = \frac{2\nu-1}{2\rho}\tilde{T}_t + \frac{1}{2\rho}\hat{F}_t$[2], we can rearrange the first rows of the equation (28) as follows:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- \left[(2\nu-1)\hat{C}_t^{R,fb} + 2\theta\nu(1-\nu)\hat{T}_t^{fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\tilde{T}_t \\
&- \frac{1}{2\nu-1}\left[(2\nu-1)\hat{C}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\hat{F}_t - 2\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Then, substituting the equations (21), (22), and (25) into the first two rows into the second row yields:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- \left(\hat{Y}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right)\tilde{T}_t \\
&- \left[\frac{(2\nu-1)}{D}\hat{Y}_t^{R,fb} - \frac{(2\nu-1)^2}{\rho D}\hat{Z}_t^R\right]\hat{F}_t - \frac{\eta}{\rho}\hat{Y}_t^{R,fb}\left[D\tilde{T}_t + (2\nu-1)\hat{F}_t\right] + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Using the equation (22) and the relation between the flexible relative price and the flexible relative output gap (10) eliminates the first two rows.

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the

[2]This relationship can be easily derived from the equations (24) and (25).

world gap variables ($\tilde{X}_t^W$), we first use the equation (7), (20) and (23) to yield:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &-\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&-\theta\nu(1-\nu)\tilde{T}_t^2 \\
&-\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (23), (24), and (25), and some rearrangement, we can derive the welfare loss function as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &(\rho+\eta)\left(\tilde{Y}_t^W\right)^2 + \left(\frac{\rho}{D}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
&+\frac{\theta\nu(1-\nu)}{D}(\hat{F}_t)^2 \\
&+\frac{\sigma}{\kappa}\left[(\hat{\pi}_t^W)^2 + (\hat{\pi}_{ppi,t}^R)^2\right].
\end{aligned}
\tag{29}
$$

2.3 Useful First- and Second-Order Conditions (LCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu-1)\hat{T}_t + \hat{m}_t$), we obtain:

$$
\hat{Y}_{H,t} = \hat{C}_t + \frac{D-(2\nu-1)}{2\rho}\hat{T}_t - \frac{1-\nu}{\rho}(\hat{m}_t + \hat{F}_t) - \frac{2(1-\nu)}{\rho}\hat{Z}_t^R, \tag{30}
$$

$$
\hat{Y}_{H,t} = \hat{C}_t^* + \frac{D+(2\nu-1)}{2\rho}\hat{T}_t + \frac{\nu}{\rho}(\hat{m}_t + \hat{F}_t) + \frac{2\nu}{\rho}\hat{Z}_t^R, \tag{31}
$$

$$
\hat{Y}_{F,t} = \hat{C}_t^* - \frac{D-(2\nu-1)}{2\rho}\hat{T}_t + \frac{1-\nu}{\rho}(\hat{m}_t + \hat{F}_t) + \frac{2(1-\nu)}{\rho}\hat{Z}_t^R. \tag{32}
$$

Then, by combining the equations (30), (31), and (32), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, currency misalignment, and the shock as follow:

$$
\hat{C}_t^W = \hat{Y}_t^W, \tag{33}
$$

$$
\hat{C}_t^R = \frac{2\nu-1}{D}\hat{Y}_t^R + \frac{2\theta\nu(1-\nu)}{D}(\hat{m}_t + \hat{F}_t) + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \tag{34}
$$

$$
\hat{T}_t = \frac{2\rho}{D}\hat{Y}_t^R - \frac{2\nu-1}{D}(\hat{m}_t + \hat{F}_t) - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{35}
$$

Adding (30) to (32) yields (33), subtracting (32) from (31) yields (35), and substituting (35) into the risk-sharing condition yields (34).

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$
\tilde{C}_t^W = \tilde{Y}_t^W, \tag{36}
$$

$$
\tilde{C}_t^R = \frac{2\nu-1}{D}\tilde{Y}_t^R + \frac{2\theta\nu(1-\nu)}{D}(\hat{m}_t + \hat{F}_t), \tag{37}
$$

$$
\tilde{T}_t = \frac{2\rho}{D}\tilde{Y}_t^R - \frac{2\nu-1}{D}(\hat{m}_t + \hat{F}_t). \tag{38}
$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \nu\Delta_{H,t}\left(\frac{P_{H,t}}{P_t}\right)^{-\theta} C_t + (1-\nu)\Delta^*_{H,t}\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\theta} C^*_t,$$
$$Y_{F,t} = (1-\nu)\Delta_{F,t}\left(\frac{P_{F,t}}{P_t}\right)^{-\theta} C_t + \nu\Delta^*_{F,t}\left(\frac{P^*_{F,t}}{P^*_t}\right)^{-\theta} C^*_t,$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\sigma} di$, $\Delta^*_{H,t} \equiv \int_0^1 \left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\sigma} di$, $\Delta_{F,t} \equiv \int_0^1 \left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\sigma} di$ and $\Delta^*_{F,t} \equiv \int_0^1 \left(\frac{P^*_{F,t}(i)}{P^*_{F,t}}\right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following:

$$\begin{aligned} 2\hat{C}^W_t - 2\hat{Y}^W_t =& -\left[\left(\hat{C}^W_t\right)^2 + \left(\hat{C}^R_t\right)^2\right] + \left[\left(\hat{Y}^W_t\right)^2 + \left(\hat{Y}^R_t\right)^2\right] \\ & - \theta\nu(1-\nu)\hat{T}^2_t \\ & - \frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}^2_{H,t} + (1-\nu)(\hat{\pi}^*_{H,t})^2 + \nu(\hat{\pi}^*_{F,t})^2 + (1-\nu)\hat{\pi}^2_{F,t}\right\}. \end{aligned} \tag{39}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (39), we get:

$$\begin{aligned} 2\tilde{C}^W_t - 2\tilde{Y}^W_t =& -\left[\left(\tilde{C}^W_t\right)^2 + \left(\tilde{C}^R_t\right)^2\right] + \left[\left(\tilde{Y}^W_t\right)^2 + \left(\tilde{Y}^R_t\right)^2\right] \\ & - \theta\nu(1-\nu)\tilde{T}^2_t \\ & - \left(\hat{C}^{fb,W}_t\tilde{C}^W_t + \hat{C}^{fb,R}_t\tilde{C}^R_t\right) + 2\left(\hat{Y}^{fb,W}_t\tilde{Y}^W_t + \hat{Y}^{fb,R}_t\tilde{Y}^R_t\right) \\ & - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\ & - \frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}^2_{H,t} + (1-\nu)(\hat{\pi}^*_{H,t})^2 + \nu(\hat{\pi}^*_{F,t})^2 + (1-\nu)\hat{\pi}^2_{F,t}\right\}. \end{aligned} \tag{40}$$

Equations (20), (21), (22), (36), (37), (38), and (40) will be used in the next subsection.

2.4 Further Manipulation of the Welfare Loss Function (LCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (40) into (13), we obtain:

$$\begin{aligned} -2\left(\mathcal{L}_t - \mathcal{L}^{fb}_t\right) =& -2\rho\hat{C}^{R,fb}_t\tilde{C}^R_t + 2\hat{Z}^R_t\tilde{C}^R_t - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\ & - 2\eta\hat{Y}^{R,fb}_t\tilde{Y}^R_t + 2(1+\eta)\hat{A}^R_t\tilde{Y}^R_t \\ & - 2\rho\hat{C}^{W,fb}_t\tilde{C}^W_t + 2\hat{Z}^W_t\tilde{C}^W_t \\ & - 2\eta\hat{Y}^{W,fb}_t\tilde{Y}^W_t + 2(1+\eta)\hat{A}^W_t\tilde{Y}^W_t \\ & - \rho\left[\left(\tilde{C}^W_t\right)^2 + \left(\tilde{C}^R_t\right)^2\right] - \eta\left[\left(\tilde{Y}^W_t\right)^2 + \left(\tilde{Y}^R_t\right)^2\right] \\ & - \theta\nu(1-\nu)\tilde{T}^2_t \\ & - \frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}^2_{H,t} + (1-\nu)(\hat{\pi}^*_{H,t})^2 + \nu(\hat{\pi}^*_{F,t})^2 + (1-\nu)\hat{\pi}^2_{F,t}\right\} + t.i.p. \end{aligned} \tag{41}$$

To recast the welfare loss in terms of output gaps, relative price gaps, currency misalignment, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R = \frac{2\nu-1}{2\rho}\tilde{T}_t + \frac{1}{2\rho}(\hat{m}_t + \hat{F}_t)$[3], we can rearrange the first two rows of the equation (41) as follows:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- \left\{(2\nu-1)\hat{C}_t^{R,fb} + 2\theta\nu(1-\nu)\hat{T}_t^{fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t \\
&- \left\{\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right\}(\hat{m}_t + \hat{F}_t) \\
&- 2\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}_{H,t}^2 + (1-\nu)(\hat{\pi}_{H,t}^*)^2\right. \\
&\qquad \left. + \nu(\hat{\pi}_{F,t}^*)^2 + (1-\nu)\hat{\pi}_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Then, substituting the equations (21) and (22) into the first row, and the equation (38) into the third row yields:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &- \left\{\hat{Y}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t \\
&- \left\{\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right\}(\hat{m}_t + \hat{F}_t) \\
&- \frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\left\{\tilde{T}_t + \frac{2\nu-1}{D}(\hat{m}_t + \hat{F}_t)\right\} + 2(1+\eta)\hat{A}_t^R\tilde{Y}_t^R \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- 2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}_{H,t}^2 + (1-\nu)(\hat{\pi}_{H,t}^*)^2\right. \\
&\qquad \left. + \nu(\hat{\pi}_{F,t}^*)^2 + (1-\nu)\hat{\pi}_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

[3] This relationship can be easily derived from the equations (37) and (38).

Using the equation (1), (2), (22) and the relation between the flexible relative price and the flexible relative output gap (10) substitutes out the first three rows.

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& -2\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W + 2(1+\eta)\hat{A}_t^W\tilde{Y}_t^W \\
& -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& -\theta\nu(1-\nu)\tilde{T}_t^2 \\
& -\frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}_{H,t}^2 + (1-\nu)(\hat{\pi}_{H,t}^*)^2\right. \\
& \qquad \left. +\nu(\hat{\pi}_{F,t}^*)^2 + (1-\nu)\hat{\pi}_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (7), (20) and (36) to yield:

$$
\begin{aligned}
-2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& -\theta\nu(1-\nu)\tilde{T}_t^2 \\
& -\frac{\sigma}{2\kappa}\left\{\nu\hat{\pi}_{H,t}^2 + (1-\nu)(\hat{\pi}_{H,t}^*)^2\right. \\
& \qquad \left. +\nu(\hat{\pi}_{F,t}^*)^2 + (1-\nu)\hat{\pi}_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (36), (37), and (38), and some rearrangement, we can derive the welfare loss function as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & (\rho+\eta)\left(\tilde{Y}_t^W\right)^2 + \left(\frac{\rho}{D}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& + \frac{\theta\nu(1-\nu)}{D}(\hat{m}_t + \hat{F}_t)^2 \\
& + \frac{\sigma}{\kappa}[(\hat{\pi}_t^W)^2 + \nu(\hat{\pi}_{ppi,t}^R)^2 + (1-\nu)(\hat{\pi}_{export,t}^R)^2] + t.i.p.
\end{aligned} \tag{42}
$$

References

Engel, Charles, "Currency Misalignments and Optimal Monetary Policy: A Reexamination," *American Economic Review*, 2011, *101* (6), 2796–2822.

Woodford, Michael, *Interest and prices*, Princeton University Press, 2003.